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Hankel Determinent for Certain Classes of Analytic Functions

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ABSTRACT:
Let A_1 denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc $E = \{z :$ |z| <1}. M_α denotes the class of functions in A_1 which satisfy the conditions $\frac{f(z).f'(z)}{z} \neq 0$ and for $0 \leq \alpha \leq 1$, $\text{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0$. We are interested in determining the sharp upper bound for the functional $\left|a_2 a_4 - a_3^2\right|$ for the class M_α .
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I. INTRODUCTION:

Let A_1 be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit disc $E = \{z :$ |z| <1 }.
S denotes the Class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

analytic and univalent in $E = \{z :$ |z| <1 }.

Let $\gamma(p)$ be the class of functions of the form

$$\text{P}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{1.3}$$

analytic in the unit disc $E = \{z :$ |z| < 1 } with *Re* $P(z) > 0$. Carathéodory [1] introduced the class $\gamma(p)$.
Noshiro [2] and Warschawiski [3] introduced the class of univalent functions

$$R = \{ f \in A_1 : \text{Re}\, f'(z) > 0, z \in E \} \tag{1.4}$$

known as N-W class of functions.
R and its subclasses were studied by several authors including Goel and Mehrok [11, 12].

$$S^* = \{ f \in A_1 : \text{Re}\frac{zf'(z)}{f(z)} > 0, z \in E \} \tag{1.5}$$

is the class of starlike univalent functions .

$$K = \{ f \in A_1 : \operatorname{Re} \frac{(zf'(z))'}{f'(z)} > 0, z \in E \} \tag{1.6}$$

is the class of convex univalent functions.

Macanu [5] introduced the class of α - convex functions defined as

$$M_\alpha = \left\{ \begin{array}{l} f \in A_1 : \frac{f(z) f'(z)}{z} \neq 0, \\ \operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0, 0 \leq \alpha \leq 1, z \in E \end{array} \right\} \tag{1.7}$$

For any real α , Miller, Macanu and Reade [7] have shown that all α - convex functions, are starlike in *E;* and for all $\alpha \geq 1$, all α -convex functions are convex in *E*.
Hassoon, Al-Amiri and Reade [9] introduced the class of analytic functions

$$H_\alpha = \left\{ f \in A_1 : \operatorname{Re}\left[(1-\alpha) f'(z) + \alpha \frac{(zf'(z))'}{f'(z)} \right] > 0, 0 \leq \alpha \leq 1, z \in E \right\} \tag{1.8}$$

Bazilevic [4] introduced the following class of analytic univalent functions. For β real, $\alpha > 0$, $P(z) \in \gamma(p)$ and $g(z) \in S^*$

$$B(\alpha, \beta, P, g) = \left\{ f \in A_1 : \left[(\alpha + i\beta) \int_0^z P(t) t^{\beta-1} g^\alpha(t) dt \right]^{\frac{1}{\alpha + t\beta}} \right\} \tag{1.9}$$

Taking $\beta = 0$ and $g(z) \equiv z$ in (1.9), we get $B(\alpha, 0, P, z)$ as the class of functions

$$B(\alpha, 0, P, z) = \left\{ f \in A_1 : \left[\alpha \int_0^z P(z) z^{\beta-1} dt \right]^{\frac{1}{\alpha}} \right\} \tag{1.10}$$

The class $B(\alpha, 0, P, z)$ was studied by Singh [6] and El-Ashwah and Thomas [13].

$$B_\alpha = \left\{ f \in A_1 : \operatorname{Re} f'(z) \left(\frac{f(z)}{z} \right)^{\alpha-1} > 0, 0 \leq \alpha \leq 1, z \in E \right\} \tag{1.11}$$

is also a subclass of Bazilevic functions.

II. Preliminary Lemmas.

Lemma 2.1[15]: Let $\mathrm{P}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \in \gamma(p)$, then $|p_n| \leq 2$ for all n (n=1,2,3...).

Lemma 2.2[9]: If $\mathrm{P}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \in \gamma(p)$, then

$2p_2 = p_1^2 + (4 - p_1^2)x$ and

$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$

for some *x* and *z* with $|x| \leq 1, |z| \leq 1$.

III. Main Results.

Theorem 3.1: Let $f \in M_\alpha$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{3\alpha(1+\alpha)^3}{(1+3\alpha)(1+2\alpha)^2\left(2+15\alpha+24\alpha^2+7\alpha^3\right)}+\frac{1}{(1+2\alpha)^2},0<\alpha\le 1; \qquad (3.1)$$

And $\left|a_2a_4-a_3^2\right|\le 1,\alpha=0$ (3.2)

Results are sharp.

Proof: Since $f\in M_\alpha$, it follows that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}=P(z) \qquad (3.3)$$

Equating the coefficients in (3.3), it is easily established that

$$\left.\begin{aligned}
a_2&=\frac{p_1}{(1+\alpha)}\\
a_3&=\frac{p_2}{2(1+3\alpha)}+\frac{(1+3\alpha)p_1^2}{2(1+2\alpha)(1+\alpha)^2}\\
a_4&=\frac{p_3}{3(1+3\alpha)}+\frac{(1+5\alpha)p_1p_2}{2(1+\alpha)(1+2\alpha)(1+3\alpha)}+\frac{\left(1+6\alpha+17\alpha^2\right)p_1^4}{6(1+2\alpha)(1+3\alpha)(1+\alpha)^3}
\end{aligned}\right\} \qquad (3.4)$$

System (3.4) yields

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}4(1+2\alpha)^2(1+\alpha)^3p_1(4p_3)+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2p_1^2(2p_2)\\+8(1+2\alpha)\left(1+6\alpha+17\alpha^2\right)p_1^4-3(1+3\alpha)\left[(1+\alpha)^2(2p_2)+2(1+3\alpha)p_1^2\right]^2\end{array}\right| \qquad (3.5)$$

$$C(\alpha)=\frac{1}{48(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4}\ . \qquad (3.6)$$

Using lemma 2.2 in (3.5), we get

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}4(1+2\alpha)^2(1+\alpha)^3p_1\left[p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]\\+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2p_1^2\left(p_1^2+\left(4-p_1^2\right)x\right)\\+8(1+2\alpha)\left(1+6\alpha+17\alpha^2\right)p_1^4-3(1+3\alpha)\left[\left(3+8\alpha+\alpha^2\right)p_1^2+(1+\alpha)^2\left(4-p_1^2\right)x\right]^2\end{array}\right| \qquad (3.7)$$

Replacing p_1 by $p\in[0,2]$, (3.7) takes the form

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}-\left[\begin{array}{l}-4(1+2\alpha)^2(1+\alpha)^3-12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-8(1+2\alpha)\left(1+6\alpha+17\alpha^2\right)+3(1+3\alpha)\left(3+8\alpha+\alpha^2\right)^2\end{array}\right]p^4\\+\left[\begin{array}{l}8(1+2\alpha)^2(1+\alpha)^3+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-6(1+3\alpha)\left(3+8\alpha+\alpha^2\right)(1+\alpha)^2\end{array}\right]p^2\left(4-p^2\right)x\\-(1+\alpha)^3\left(4-p^2\right)\left[12(1+\alpha)(1+3\alpha)+\left(1+4\alpha+7\alpha^2\right)\right]x^2\\+8(1+2\alpha)^2(1+\alpha)^3p\left(4-p^2\right)\left(1-|x|^2\right)z\end{array}\right| \qquad (3.8)$$

Applying triangular inequality to (3.8), we obtain

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}-4(1+2\alpha)^2(1+\alpha)^3-12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-8(1+2\alpha)(1+6\alpha+17\alpha^2)+3(1+3\alpha)(3+8\alpha+\alpha^2)^2\end{array}\right)p^4\\+\left(\begin{array}{l}8(1+2\alpha)^2(1+\alpha)^3+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2\end{array}\right)p^2(4-p^2)|x|\\+(1+\alpha)^3(4-p^2)(2-p)\left(6(1+\alpha)(1+3\alpha)-(1+4\alpha+7\alpha^2)\right)|x|^2\\+\left(8(1+2\alpha)^2(1+\alpha)^3\right)p(4-p^2)\end{array}\right] \quad (3.9)$$

$$=\frac{1}{C(\alpha)}F(\sigma),\sigma=|x|\le 1. \quad (3.10)$$

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and $F(\sigma)$ attains its maximum value at $|\sigma|=|x|=1$.

Putting $|x|=1$ in (3.9) , we have

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}-4(1+2\alpha)^2(1+\alpha)^3-12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-8(1+2\alpha)(1+6\alpha+17\alpha^2)+3(1+3\alpha)(3+8\alpha+\alpha^2)^2\end{array}\right)p^4\\+\left(\begin{array}{l}8(1+2\alpha)^2(1+\alpha)^3+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2\end{array}\right)p^2(4-p^2)\\+(1+\alpha)^3(4-p^2)\left(12(1+\alpha)(1+3\alpha)+(1+4\alpha+7\alpha^2)p^2\right)\end{array}\right]$$

$$=\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}3(1+3\alpha)(3+8\alpha+\alpha^2)^2+6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2-12(1+2\alpha)^2(1+\alpha)^3\\-24(1+2\alpha)(1+5\alpha)(1+\alpha)^2-8(1+2\alpha)(1+6\alpha+17\alpha^2)-(1+\alpha)^3(1+4\alpha+7\alpha^2)\end{array}\right)p^4\\+4\left(\begin{array}{l}8(1+2\alpha)^2(1+\alpha)^3+12(1+2\alpha)(1+5\alpha)(1+\alpha)^2\\-6(1+3\alpha)(3+8\alpha+\alpha^2)(1+\alpha)^2+4(1+\alpha)^3(1+4\alpha+7\alpha^2)-12(1+3\alpha)(1+\alpha)^4\\+4(1+\alpha)^3(1+3\alpha+7\alpha^2)\end{array}\right)p^2\\+48(1+3\alpha)(1+\alpha)^4\end{array}\right]$$

$$\left.\begin{array}{l}=\dfrac{1}{C(\alpha)}\left[-A(\alpha)p^4+B(\alpha)p^2+48(1+3\alpha)(1+\alpha)^4\right]\\=\dfrac{1}{C(\alpha)}G(p)\end{array}\right] \quad (3.11)$$

where $A(\alpha)=4\alpha(1+\alpha)(7\alpha^3+24\alpha^2+15\alpha+2)$, (3.12)

and $B(\alpha)=48\alpha(1+\alpha)^4$. (3.13)

$$G'(p)=-4A(\alpha)p^3+2B(\alpha)p=0 \quad (3.14)$$

which implies that $p=0$ or $p^2=\dfrac{B(\alpha)}{2A(\alpha)}$.

p=0 does not give maximum value and is rejected.

$$p^2 = \frac{B(\alpha)}{2A(\alpha)} = \frac{6(1+\alpha)^3}{(2+15\alpha+24\alpha^2+7\alpha^3)}, \quad (\alpha \neq 0). \tag{3.15}$$

gives the maximum value of $G(p)$.

Putting the value of p^2 from (3.15) in (3.11) , we get

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)}\left[\frac{B^2(\alpha)}{4A(\alpha)} + 48(1+3\alpha)(1+\alpha)^4\right], \ \alpha \neq 0 \tag{3.16}$$

Substituting the values from (3.6), (3.12) and (3.13) in (3.16) , the bound (3.1) follows.
Consider the case $\alpha = 0$. In this case, $A(\alpha)=0, B(\alpha)=0$ and $C(\alpha)=48$.
Putting these values in (3.11) , we get $|a_2a_4 - a_3^2| \leq 1$.

Result (3.1) is sharp for $p_1 = \sqrt{\frac{6(1+\alpha)^3}{(2+15\alpha+24\alpha^2+7\alpha^3)}}$, $p_2 = -1$ and p_3 obtained from (3.5).
Result (3.2) is sharp for $p_1 = 0, p_2 = -1$ and $p_3 = -2$.

Remark 3.1: Taking $\alpha = 1$ in (3.1) , we get $|a_2a_4 - a_3^2| \leq \frac{1}{8}$, a result due to Janteng etal.[15].
Remark 3.2: Result (3.2) is also due to Janteng etal.[15].
Theorem 3.2: Let $f \in H_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}\frac{1}{(1+\alpha)^2}, 0 \leq \alpha \leq \frac{5}{17} \ ; \tag{3.17}$$

and $$|a_2a_4 - a_3^2| \leq \frac{(17\alpha-5)^2}{144(1+2\alpha)(1+20\alpha+7\alpha^2-4\alpha^3)}, + \frac{4}{9(1+\alpha)^2}, \frac{5}{17} \leq \alpha \leq 1. \tag{3.18}$$

Results are sharp.

Proof: Since $f \in H_\alpha$, therefore by definition $(1-\alpha)f'(z) + \alpha\frac{(zf'(z))'}{f'(z)} = P(z)$.

Identification of terms in the above equation yields

$$\left.\begin{aligned} a_2 &= \frac{p_1}{2} \\ a_3 &= \frac{(p_2+\alpha p_1^2)}{3(1+\alpha)} \\ a_4 &= \frac{p_3}{4(1+2\alpha)} + \frac{3\alpha p_1p_2}{4(1+\alpha)(1+2\alpha)} + \frac{\alpha(2\alpha-1)p_1^3}{4(1+\alpha)(1+2\alpha)} \end{aligned}\right\} \tag{3.19}$$

From (3.19), we obtain

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)}\left|\begin{matrix} 9(1+\alpha)^2 p_1(4p_3) + 54\alpha(1+\alpha)p_1^2(2p_2) \\ +36\alpha(2\alpha-1)(1+\alpha)p_1^4 - 8(1+2\alpha)(2p_2+2\alpha p_1^2)^2 \end{matrix}\right| , \tag{3.20}$$

Where $C(\alpha)=\frac{1}{288(1+2\alpha)(1+\alpha)^2}$. (3.21)

By lemma 2.2, we get

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}9(1+\alpha)^2p_1\left[p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]\\+54(1+\alpha)p_1^2\left[p_1^2+\left(4-p_1^2\right)x\right]-8(1+2\alpha)\left[(1+2\alpha)p_1^2+\left(4-p_1^2\right)x\right]^2\\+36\alpha(2\alpha-1)(1+\alpha)p_1^4\end{array}\right| \quad (3.22)$$

$|x|\le 1$ and $|z|\le 1$. Changing p_1 to $p\in[0,2]$, (3.22) takes the form

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}\left(1-12\alpha+3\alpha^2+8\alpha^3\right)p^4+2\left(1+13\alpha+4\alpha^2\right)p^2\left(4-p^2\right)x\\-\left(4-p^2\right)\left[32(1+2\alpha)+\left(1+2\alpha+9\alpha^2\right)p^2\right]\\+18(1+\alpha)^2p\left(4-p^2\right)\left(1-|x|^2\right)z\end{array}\right|$$

$$\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(1-12\alpha+3\alpha^2+8\alpha^3\right)p^4+2\left(1+13\alpha+4\alpha^2\right)p^2\left(4-p^2\right)\sigma\\+\left(4-p^2\right)(2-p)\left[16(1+2\alpha)-\left(1+2\alpha+9\alpha^2\right)p^2\right]\sigma^2\\+18(1+\alpha)^2p\left(4-p^2\right)\end{array}\right]=\frac{1}{C(\alpha)}F(\sigma),\ \sigma=|x|\le 1. \quad (3.23)$$

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and maximum $F(\sigma)=F(1)$.

Putting the value $\sigma=1$ in (3.23), we arrive at

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(1-12\alpha+3\alpha^2+8\alpha^3\right)p^4+2\left(1+13\alpha+4\alpha^2\right)p^2\left(4-p^2\right)\\+\left(4-p^2\right)\left(32(1+2\alpha)+\left(1+2\alpha+9\alpha^2\right)p^2\right)\end{array}\right]$$

$$=\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\left(1-12\alpha+3\alpha^2+8\alpha^3\right)-2\left(1+13\alpha+4\alpha^2\right)-\left(1+2\alpha+9\alpha^2\right)\right)p^4\\+\left[8\left(1+13\alpha+\alpha^2\right)-32(1+2\alpha)+4\left(1+2\alpha+9\alpha^2\right)\right]p^2+128(1+2\alpha)\end{array}\right]$$

$$\left.\begin{array}{l}=\frac{1}{C(\alpha)}\left[-A(\alpha)p^4+B(\alpha)p^2+128(1+2\alpha)\right]\\=\frac{1}{C(\alpha)}G(p)\end{array}\right\} \quad (3.24)$$

Where $A(\alpha)=2\left(1+20\alpha+7\alpha^2-9\alpha^3\right)>0$ in [0, 1] (3.25)

And $B(\alpha)=4(1+\alpha)(17\alpha-5)$. (3.26)

Case I: $0\le\alpha\le\frac{5}{17}$ so that $B(\alpha)\le 0$.

$G'(p)<0$ and $G(p)$ attains its maximum value at $p=0$.

From (3.21) and (3.24) it follows that $\left|a_2a_4-a_3^2\right|\le\frac{4}{9}\frac{1}{(1+\alpha)^2}$.

Result is sharp for $p_1=0, p_2=-1$ and $p_3=-2$.

Case II: $\frac{5}{17} \le \alpha \le 1$ so that $B(\alpha) > 0$.

$G'(p) = -4A(\alpha)p^3 + 2B(\alpha)p = 0$ which implies that

$$p = 0 \text{ or } p^2 = \frac{B(\alpha)}{2A(\alpha)} = \frac{(1+\alpha)(17\alpha - 5)}{(1+20\alpha+7\alpha^2-4\alpha^3)} \quad (3.27)$$

$p = 0$ does not give the maximum value and is rejected.

Substituting the value of p^2 from (3.27) in (3.24), we conclude that

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}\left[\frac{B^2(\alpha)}{4A(\alpha)} + 128(1+2\alpha)\right] \quad (3.28)$$

Putting the values from (3.21), (3.25) and (3.26) in (3.28), result (3.18) follows.

Equality sign in (3.18) holds for $p_1 = \sqrt{\frac{(1+\alpha)(17\alpha-5)}{(1+20\alpha+7\alpha^2-4\alpha^3)}}$, $p_2 = -1$ and p_3 obtained from (3.20).

Remark 3.3: Letting $\alpha = 0$ in (3.17) , we get

$\left|a_2a_4 - a_3^2\right| \le \frac{4}{9}$, a result proved by Janteng etal.[7] for the class *R*.

Remark 3.4: Letting $\alpha = 1$ in (3.18), it follows that

$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8}$, result proved by Janteng etal.[8] for the class *K*

Theorem 3.3: Let $f \in B_\alpha$, then $\left|a_2a_4 - a_3^2\right| \le \frac{4}{(\alpha+2)^2}$. (3.29)

The result is sharp.

Proof: Since $f \in B_\alpha$, therefore it follows that

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} = P(z), (0 \le \alpha \le 1). \quad (3.30)$$

Equating the coefficients in (3.30) , we get

$$\left.\begin{aligned} a_2 &= \frac{p_1}{(\alpha+1)} \\ a_3 &= \frac{p_2}{(\alpha+2)} - \frac{(\alpha-1)p_1^2}{2(\alpha+1)^2} \\ a_4 &= \frac{p_3}{(\alpha+3)} - \frac{(\alpha-1)p_1p_2}{(\alpha+1)(\alpha+2)} + \frac{(\alpha-1)(2\alpha-1)p_1^3}{6(\alpha+1)^3} \end{aligned}\right\} \quad (3.31)$$

(3.31) gives

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)}\left|\begin{aligned} &3(\alpha+2)^2(\alpha+1)^3 p_1(4p_3) - 6(\alpha-1)(\alpha+2)(\alpha+1)^2 p_1(2p_2) \\ &+ 2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2 p_1^4 - 3(\alpha+3)\left[2(1+\alpha)p_2 - (\alpha-1)(\alpha+2)p_1^2\right]^2 \end{aligned}\right| \quad (3.32)$$

Where $C(\alpha) = \frac{1}{12(\alpha+3)(\alpha+2)^2(\alpha+1)^4}$. (3.33)

Using lemma 2.2 in (3.32), we obtain

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}3(\alpha+2)^2(\alpha+1)^3p_1\left[p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z\right]\\-6(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2p_1^2\left[p_1^2+\left(4-p_1^2\right)x\right]\\+2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2p_1^4-3(\alpha+3)\left[(\alpha+3)p_1^2+(1+\alpha)^2\left(4-p_1^2\right)x\right]^2\end{array}\right| \quad (3.34)$$

Changing p_1 to $p\in[0,2]$, (3.34) takes the form

$$\left|a_2a_4-a_3^2\right|=\frac{1}{C(\alpha)}\left|\begin{array}{l}\left[\begin{array}{l}3(\alpha+2)^2(\alpha+1)^3-6(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2\\+2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2-3(\alpha+3)^3\end{array}\right]p^4\\+\left[\begin{array}{l}6(\alpha+2)^2(\alpha+1)^3-6(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2\\-6(\alpha+1)^2(\alpha+3)^2\end{array}\right]p^2\left(4-p^2\right)x\\-3(\alpha+1)^3\left(4-p^2\right)\left[4(\alpha+1)(\alpha+3)+p^2\right]x^2\\+6(\alpha+2)^2(\alpha+1)^3p\left(4-p^2\right)\left(1-|x|^2\right)z\end{array}\right| \quad (3.35)$$

Coefficient of p^4 in (3.35) changes from negative to positive in [0, 1] and therefore there must exist α_0 in (0, 1) so that coefficient of p^4 is negative for $0\le\alpha<\alpha_0$ and positive for $\alpha_0\le\alpha\le1$.

Case I: $0\le\alpha<\alpha_0$, from (3.35) it follows that

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}3(\alpha+3)^3+6(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2-3(\alpha+2)^2(\alpha+1)^3\\-2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2\end{array}\right)p^4\\+6\left(\begin{array}{l}(\alpha+2)^2(\alpha+1)^3-(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2\\-(\alpha+1)^2(\alpha+3)^2\end{array}\right)p^2\left(4-p^2\right)x\\+3(\alpha+1)^3\left(4-p^2\right)(2-p)\left[2(\alpha+1)(\alpha+3)-p\right]x^2\\+6(\alpha+2)^2(\alpha+1)^3p\left(4-p^2\right)\end{array}\right]$$

$$=\frac{1}{C(\alpha)}F(\sigma),\sigma=|x|\le1\,. \quad (3.36)$$

$F'(\sigma)>0$ and therefore $F(\sigma)$ is increasing in [0,1] and $F(\sigma)$ will attain its maximum value at $\sigma=|x|=1$.

Putting $\sigma=|x|=1$ in (3.36), we get

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{C(\alpha)}\left[\begin{array}{l}\left(\begin{array}{l}3(\alpha+3)^3+6(\alpha+3)^2(\alpha+1)^2+12(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2\\-9(\alpha+2)^2(\alpha+1)^3-2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2-3(\alpha+1)^3\end{array}\right)p^4\\+12\left(\begin{array}{l}2(\alpha+2)^2(\alpha+1)^3-2(\alpha-1)(\alpha+2)(\alpha+3)(\alpha+1)^2\\-2(\alpha+1)^2(\alpha+3)^2-(\alpha+1)^3\left[(\alpha+1)(\alpha+3)+1\right]\end{array}\right)p^2\left(4-p^2\right)\\+48\alpha(\alpha+3)(\alpha+1)^4\end{array}\right]$$

$$\left.\begin{aligned} &= \frac{1}{C(\alpha)}\left[-A_1(\alpha)p^4 - B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4\right] \\ &= \frac{1}{C(\alpha)}G(p) \end{aligned}\right] . \tag{3.37}$$

Where $$\left.\begin{aligned} A_1(\alpha) &= \alpha(\alpha+1)\left(\alpha^3 + 6\alpha^2 + 21\alpha + 20\right) > 0 \\ B(\alpha) &= 12\alpha(\alpha+4)(\alpha+1)^3 > 0 \end{aligned}\right] . \tag{3.38}$$

$G'(p) < 0$, $G(p)$ is decreasing in [0, 2] and maximum $G(p) = G(0) = 48(1+3\alpha)(\alpha+1)^4$.
From (3.37) and (3.33), it follows that

$$\left|a_2 a_4 - a_3^2\right| \le \frac{4}{(\alpha+2)^2}.$$

Case II: $\alpha_0 \le \alpha \le 1$, proceeding as in case I, from (3.35) , we get

$$\left|a_2 a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}\left[\begin{matrix} \left[\begin{matrix} \left(6(\alpha+2)^2(\alpha+1)^2 + 2(\alpha-1)(2\alpha-1)(\alpha+3)(\alpha+2)^2\right) \\ -3\left((\alpha+2)^2(\alpha+1)^3 - (\alpha+3)^3 - (\alpha+1)^3\right) \end{matrix}\right] p^4 \\ -B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4 \end{matrix}\right].$$

Where $B(\alpha)$ is given by (3.38))

$$= \frac{1}{C(\alpha)}\left[-A_2(\alpha)p^4 - B(\alpha)p^2 + 48(1+3\alpha)(1+\alpha)^4\right]. \tag{3.39}$$

where $A_2(\alpha) = \left[18 + 34\alpha + 13\alpha^2 - 4\alpha^3 - 7\alpha^4 - \alpha^5\right] > 0$.

As discussed in case I, $\left|a_2 a_4 - a_3^2\right| \le \dfrac{4}{(\alpha+2)^2}$.

Combining both the cases, proof of the theorem is complete.
Result (3.29) is sharp for $p_1 = 0$, $p_2 = -1$ and $p_3 = -2$.

Remark 3.5: Putting $\alpha = 0$ in (3.29) , we have
$\left|a_2 a_4 - a_3^2\right| \le 1$, a result established by Janteng etal.[15] for the class S^*.

Remark 3.6: If we put $\alpha = 1$ in (3.29), we get
$\left|a_2 a_4 - a_3^2\right| \le \dfrac{4}{9}$, a result established by Janteng etal.[14] for the class *R*.

References

[1] C. Carathéodory, Über Den Variabilitatsb Erecöch Der Fourieŕ Schen Konstanten Von Positivenen harmonischen-Funktionan, Rendiconti Palerno, 32(1911), 193-217.
[2] K.Noshiro, On the theory of schlicht functions,J. Fac. Soc. Hokkaido univ.Ser.1,2(1934-1935), 129-155.
[3] S. Warchawski, On the higher derivative at the boundary in conformal mapping, Trasaction Amer. Math.Soc.,38(1935), 310-340.
[4] I.E.Bazilevic, On the case of integrability in quadrature of the Lowner equation, Mat Sb. 37(1958), 471-476.
[5] P.T.Mocanu , Une propritéde convexité generalise dans la théorie de la representation conforme,Mathematica(CLUJ), 34 (1969), 127-133.
[6] R.Singh, On Bazilevic functions, Proceeding Amer. Math.Soc.,38(1973), 261-263.
[7] S.S.Miller ,P.T.Mocanu and M.O.Reade, All α -convex functions are univalent and starlike, Proceeding Amer. Math.Soc.,37(1973), 553-554.
[8] Ch. Pommerenke, Univalent functions, Vandenhoeck and Ruprecht, Gottingen,1975.

[9] S.Hassoon,Al-Amiri and M.O.Reade, On A linear combination of some expressions in the theory of the univalent functions, Monatshafte fur Math.,80(1975),257-264.
[10] R.J.Libera and E.J.Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proceeding Amer. Math.Soc., 85(1982) , 225-230.
[11] R.M.Goel and B.S.Mehrok, A subclass of univalent functions, Houston J.Math.,8(1982),343-357.
[12] R. M. Goel, Beant Singh Mehrok, A subclass of univalent functions, J.Austral.Math.Soc. (Series A),35(1983), 1-17.
[13] R.M.El-Ashwah and D.K.Thomas, Growth results for a subclass of Bazilevic functions, Inter J.Math.ana Math Sci, 8(1985), 785-792.
[14] A.Janteng , S.A.Halim and M.Darus, Coefficient inequality for a function whoose derivative has a positive real part, J.Ineq. in pure and appl. Math., 7(2006), 1-4.
[15] Aini Janteng, Hankel determinant for starlike and convex functions, Int. J.Math.Anal., 1(2007) , 619-625.